THE ADLER-SHIOTA-VAN MOERBEKE FORMULA

FOR THE BKP HIERARCHY

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Abstract

We study the BKP hierarchy and prove the existence of an Adler–Shiota–van Moerbeke formula. This formula relates the action of the $BW_{1+\infty}$–algebra on tau–functions to the action of the "additional symmetries" on wave functions.

1. Introduction and main result

1.1. Adler, Shiota and van Moerbeke [ASV1-2] obtained for the KP and Toda lattice hierarchies a formula which translates the action of the vertex operator on tau–functions to an action of a vertex operator of pseudo-differential operators on wave functions. This relates the additional symmetries of the KP and Toda lattice hierarchy to the $W_{1+\infty}$–, respectively $W_{1+\infty}\times W_{1+\infty}$–algebra symmeties. In this paper we investigate the existence of such an Adler–Shiota–van Moerbeke formula for the BKP hierarchy.
1.2. The BKP hierarchy is the set of deformation equations

$$\frac{\partial L}{\partial t_k}=[(L^n)_+,L],\quad k=3,5,\dots$$

for the first order pseudo-differential operator

$$L\equiv L(x,t)=\partial+u_1(x,t)\partial^{-1}+u_2(x,t)\partial^{-2}+\cdots,$$

here $\partial=\frac{\partial}{\partial x}$ and $t=(t_3,t_5,\dots)$. It is well–known that L dresses as $L=P\partial P^{-1}$ with

$$\begin{aligned}P\equiv P(x,t)&=1+a_1(x,t)\partial^{-1}+a_2(x,t)\partial^{-2}+\cdots\\&=\frac{\tau(x-2\partial^{-1},t-2[\partial^{-1}])}{\tau(x,t)},\end{aligned}$$

where τ is the famous τ–function, introduced by the Kyoto group [DJKM1-3] and $[z]=(\frac{z^3}{3},\frac{z^5}{5},\dots)$.

The wave or Baker–Akhiezer function

$$w\equiv w(,x,t,z)=W(x,t,\partial)e^{xz},$$

where

$$W\equiv W(x,t,z)=P(x,t)e^{\xi(x,t,z)}\qquad\text{with}\quad \xi(x,t,z)=\sum_{k=1}^{\infty}t_{2k+1}\partial^{2k+1}$$

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is an eigenfunction of L, viz.,

$$Lw = zw \qquad \text{and} \quad \frac{\partial w}{\partial t_k} = (L^k)_+ w.$$

Now introduce, following Orlov and Schulman [OS], the pseudo-differential operator $M \equiv M(x,t) = WxW^{-1}$ which action on w amounts to

$$Mw = \frac{\partial w}{\partial z},$$

then $[L, M] = 1$ and

$$\frac{\partial M}{\partial t_k} = [(L^n)_+, M], \quad k = 3, 5, \ldots$$

Let

$$Y(y,w) = \sum_{\ell=0}^{\infty} \frac{(y-w)^\ell}{\ell!} \sum_{k \in \mathbf{Z}} w^{-k-\ell-1}(M^\ell L^{k+\ell} - (-)^{k+\ell} L^{k+\ell-1} M^\ell L), \tag{1.1}$$

then one has the following main result

Theorem 1.1.

$$2(w-y)Y(y,w)_- w(x,t,z) = (w+y)(e^{-\eta(x,t,z)} - 1)\left(\frac{X(y,w)\tau(x,t)}{\tau(x,t)}\right) w(x,t,z), \tag{1.2}$$

where $X(y,w)$ is the following vertex operator

$$X(y,w) = w^{-1} \exp(x(y-w) + \sum_{j>2,\text{odd}} t_j(y^j - w^j) \exp(-2\frac{\partial}{\partial x}(y^{-1} - w^{-1}) - 2\sum_{j>2,\text{odd}} \frac{\partial}{\partial t_j} \frac{y^{-j} - w^{-j}}{j}). \tag{1.3}$$

Formula (1.2) is the Adler–Shiota–van Moerbeke formula for the BKP hierarchy, we will give a proof of this formula in section 6. This formula relates the "additional symmetries" of the BKP hierarchy, generated by $Y(y,w)$, to the $BW_{1+\infty}$–algebra, generated by $X(y,w)$. This $BW_{1+\infty}$–algebra is a subalgebra of $W_{1+\infty}$, which is defined as the -1–eigenspace of an anti–involution on $W_{1+\infty}$.

2. The Lie algebras o_∞, B_∞ and $BW_{1+\infty}$

2.1. Let $\overline{gl_\infty}$ be the Lie algebra of complex infinite dimensional matrices such that all nonzero entries are within a finite distance from the main diagonal, i.e.,

$$\overline{gl_\infty} = \{(a_{ij})_{i,j\in\mathbf{Z}} | g_{ij} = 0 \text{ if } |i-j| >> 0\}.$$

The elements E_{ij}, the matrix with the (i,j)–th entry 1 and 0 elsewhere, for $i,j \in \mathbf{Z}$ form a basis of a subalgebra $gl_\infty \subset \overline{gl_\infty}$. The Lie algebra $\overline{gl_\infty}$ has a universal central extension $A_\infty = \overline{gl_\infty} \oplus \mathbf{C}c_A$ with the Lie bracket defined by

$$[a + \alpha c_A, b + \beta c_A] = ab - ba + \mu(a,b)c_A, \tag{2.1}$$

for $a, b \in \overline{gl_\infty}$ and $\alpha, \beta \in \mathbf{C}$; here μ is the following 2–cocycle:

$$\mu(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}(\theta(i) - \theta(j)), \tag{2.2}$$

where the function $\theta : \mathbf{R} \to \mathbf{R}$ is defined by

$$\theta(i) = \begin{cases} 0 & \text{if } i > 0, \\ 1 & \text{if } i \le 0. \end{cases} \tag{2.3}$$

The Lie algebra gl_∞ and $\overline{gl_\infty}$ both have a natural action on the space of column vectors, viz., let $\mathbf{C}^\infty = \bigoplus_{k\in\mathbf{Z}} e_k$, then $E_{ij}e_k = \delta_{jk}e_i$. By identifying e_k with t^{-k}, we can embed the algebra $\mathbf{D}$ of differential operators on the circle, with basis $-t^{j+k}(\frac{\partial}{\partial t})^k$ $(j \in \mathbf{Z}, k \in \mathbf{Z}_+)$, in $\overline{gl_\infty}$:

$$\begin{aligned} &\rho : \mathbf{D} \to \overline{gl_\infty}, \\ &\rho(-t^{j+k}(\frac{\partial}{\partial t})^k) = \sum_{m\in\mathbf{Z}} -m(m-1)\cdots(m-k+1)E_{-m-j,-m}. \end{aligned} \tag{2.4}$$

It is straightforward to check that the 2–cocycle μ on $\overline{gl_\infty}$ induces the following 2–cocycle on $\mathbf{D}$:

$$\mu(-t^{i+j}(\frac{\partial}{\partial t})^j, -t^{k+\ell}(\frac{\partial}{\partial t})^\ell) = \delta_{i,-k}(-)^j j!\ell! \binom{i+j}{j+\ell+1}. \tag{2.5}$$

This cocycle was discovered by Kac and Peterson in [KP] (see also [R], [KR]). In this way we have defined a central extension of $\mathbf{D}$, which we denote by $W_{1+\infty} = \mathbf{D} \oplus \mathbf{C}c_A$, the Lie bracket on $W_{1+\infty}$ is given by

$$\begin{aligned} &[-t^{i+j}(\frac{\partial}{\partial t})^j + \alpha c_A, -t^{k+\ell}(\frac{\partial}{\partial t})^\ell + \beta c_A] = \\ &\sum_{m=0}^{\max(j,\ell)} m!(\binom{i+j}{m}\binom{\ell}{m} - \binom{k+\ell}{m}\binom{j}{m})(-t^{i+j+k+\ell-m}(\frac{\partial}{\partial t})^{j+\ell-m}) + \delta_{i,-k}(-)^j j!\ell!\binom{i+j}{j+\ell+1}c_A. \end{aligned} \tag{2.6}$$

Let $D = t\frac{\partial}{\partial t}$, then we can rewrite the elements $-t^{i+j}(\frac{\partial}{\partial t})^j$, viz. ,

$$-t^{i+j}(\frac{\partial}{\partial t})^j = -t^i D(D-1)(D-2)\cdots(D-j+1). \tag{2.7}$$

Then

$$\rho(t^k f(D) = \sum_{j\in\mathbf{Z}} f(-j)E_{j-k,j}, \tag{2.8}$$

and the 2–cocycle is as follows [KR]:

$$\mu(t^k f(D), t^\ell g(D)) = \begin{cases} \sum_{-k\le j\le -1} f(j)g(j+k) & \text{if } k = -\ell \ge 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

hence the bracket is

$$[t^k f(D), t^\ell g(D)] = t^{k+\ell}(f(D+\ell)g(D) - g(D+k)f(D)) + \mu(t^k f(D), t^\ell g(D)). \tag{2.10}$$

2.2. Define on $\overline{gl_\infty}$ the following linear anti–involution:

$$\iota(E_{jk}) = (-)^{j+k} E_{-k,-j}. \tag{2.11}$$

Using this anti–involution we define the Lie algebra $\overline{o_\infty}$ as a subalgebra of $\overline{gl_\infty}$:

$$\overline{o_\infty} = \{a \in \overline{gl_\infty} | \iota(a) = -a\}. \tag{2.12}$$

The elements $F_{jk} = E_{-j,k} - (-)^{j+k} E_{-k,j} = -(-)^{j+k} F_{kj}$ with $j < k$ form a basis of $o_\infty = \overline{o_\infty} \cap gl_\infty$. The 2–cocycle μ on $\overline{gl_\infty}$ induces a 2–cocycle on $\overline{o_\infty}$, and hence we can define a central extension $B_\infty = \overline{o_\infty} \oplus \mathbf{C}c_B$ of $\overline{o_\infty}$, with Lie bracket

$$[a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2}\mu(a,b)c_B, \tag{2.13}$$

for $a, b \in \overline{o_\infty}$ and $\alpha, \beta \in \mathbf{C}$. It is then straightforward to check that the anti–involution ι induces

$$\iota(t) = -t, \quad \iota(D) = -D. \tag{2.14}$$

Hence, it induces the following anti–involution on $\mathbf{D}$:

$$\iota(t^k f(D)) = f(-D)(-t)^k. \tag{2.15}$$

Define $\mathbf{D}^B$=$\mathbf{D} \cap \overline{o_\infty} = \{w \in \mathbf{D} | \iota(w) = -w\}$, it is spanned by the elements

$$W_k(f) := t^k f(D) - f(-D)(-t)^k = t^k(f(D) - (-)^k f(-D-k)). \tag{2.16}$$

It is straightforward to check that

$$\rho(W_k(f)) = \sum_{j \in \mathbf{Z}} f(-j) F_{k-j,j}.$$

The restriction of the 2–cocycle μ on $\mathbf{D}$, given by (2.5) or (2.9), induces a 2–cocycle on $\mathbf{D}^B$, which we shall not calculate explicitly here. It defines a central extension $BW_{1+\infty} = \mathbf{D}^B \oplus \mathbf{C}c_B$ of $\mathbf{D}^B$, with Lie bracket

$$[a + \alpha c_B, b + \beta c_B] = ab - ba + \frac{1}{2}\mu(a,b)c_B,$$

for $a, b \in \mathbf{D}^B$ and $\alpha, \beta \in \mathbf{C}$.

3. The spin module

3.1. We now want to consider highest weight representations of o_∞, B_∞ and $BW_{1+\infty}$. For this purpose we introduce the Clifford algebra BCl as the associative algebra on the generators ϕ_j, $j \in \mathbf{Z}$, called *neutral free fermions*, with defining relations

$$\phi_i\phi_j + \phi_j\phi_i = (-)^i\delta_{i,-j}. \tag{3.1}$$

We define the spin module V over BCl as the irreducible module with highest weight vector the *vacuum vector* $|0>$ satisfying

$$\phi_j|0>= 0 \quad \text{for } j > 0. \tag{3.2}$$

The elements $\phi_{j_1}\phi_{j_2}\cdots\phi_{j_p}|0>$ with $j_1 < j_2 < \cdots < j_p \le 0$ form a basis of V. Then

$$\begin{aligned} \pi(F_{jk}) &= \frac{(-)^j}{2}(\phi_j\phi_k - \phi_k\phi_j), \\ \hat\pi(F_{jk}) &= (-)^j : \phi_j\phi_k :, \\ \hat\pi(c_B) &= I, \end{aligned} \tag{3.3}$$

where the normal ordered product $:\quad:$ is defined as follows

$$:\phi_j\phi_k := \begin{cases} \phi_j\phi_k & \text{if } k > j, \\ \frac{1}{2}(\phi_j\phi_k - \phi_k\phi_j) & \text{if } j = k, \\ -\phi_k\phi_j & \text{if } k < j, \end{cases} \tag{3.4}$$

define representations of o_∞, respectively B_∞.

When restricted to o_∞ and B_∞, the spin module V breaks into the direct sum of two irreducible modules. To describe this decomposition we define a $\mathbf{Z}_2$–gradation on V by introducing a chirality operator χ satisfying $\chi|0>=|0>,\ \chi\phi_j + \phi_j\chi = 0$ for all $j \in \mathbf{Z}$, then

$$V = \bigoplus_{\alpha\in\mathbf{Z}_2} V_\alpha \quad \text{where } V_\alpha = \{v \in V|\chi v = (-)^\alpha v\}.$$

Each module V_α is an irreducible highest weight module with highest weight vector $|0>$, $|1>=\sqrt{2}\phi_0|0>$ for $V_0,\ V_1$, respectively, in the sense that $\hat\pi(c_B) = 1$ and

$$\begin{aligned} \pi(F_{-i,j})|\alpha> = \hat\pi(F_{-i,j})|\alpha>&= 0 \quad \text{for } i < j, \\ \pi(F_{-i,i})|\alpha> &= -\frac{(-)^i}{2}|\alpha> \quad \text{for } i > 0, \\ \hat\pi(F_{-i,i})|\alpha> &= 0. \end{aligned} \tag{3.5}$$

Clearly V_α is also a highest weight module for $BW_{1+\infty}$, viz

$$\hat\pi\cdot\rho(W_k(f)) = \sum_{j\in\mathbf{Z}}(-)^{k+j}f(-j) : \phi_{k-j}\phi_j :, \tag{3.6}$$

and

$$\hat\pi\cdot\rho(W_k(f))|\alpha>= 0 \quad \text{for } k \ge 0. \tag{3.7}$$

From now on we will omit o_∞, $\hat\pi$ and $\hat\pi\cdot\rho$, whenever no confusion can arise.

4. Vertex operators

4.1. Using the boson fermion correspondence (see e.g. [DJKM 3], [K], [tKL] and [Y]), we can express the fermions in terms of differential operators, i.e. there exists an isomorfism $\sigma : V \to \mathbf{C}[\theta, t_1, t_3, \cdots]$, where $\theta^2 = 0$, $t_i t_j = t_j t_i$, $\theta t_j = t_j \theta$ and $V_\alpha = \theta^\alpha \mathbf{C}[t_1, t_3, \cdots]$. such that $\sigma(|0>) = 1$. Define the following two generating series (fermionic fields):

$$\phi^\pm(z) = \sum_{j\in\mathbf{Z}} \phi_j^\pm z^{-j} = \sum_{j\in\mathbf{Z}} (\pm)^j \phi_j z^{-j}, \tag{4.1}$$

then one has the following vertex operator for these fields:

$$\sigma\phi^\pm(z)\sigma^{-1} = \frac{\theta + \frac{\partial}{\partial\theta}}{\sqrt{2}} \exp(\pm \sum_{j>0,\mathrm{odd}} t_j z^j) \exp(\mp 2 \sum_{j>0,\mathrm{odd}} \frac{\partial}{\partial t_j} \frac{z^{-j}}{j}). \tag{4.2}$$

4.2. Define

$$\begin{aligned} W(y,w) &= \sum_{\ell=0}^\infty \frac{(y-w)^\ell}{\ell!} W^{(\ell+1)}(w) \\ &= \sum_{\ell=0}^\infty \frac{(y-w)^\ell}{\ell!} \sum_{k\in\mathbf{Z}} W_k^{(\ell+1)} w^{-k-\ell-1} \\ &:= \frac{:\phi^+(y)\phi^-(w):}{w}, \end{aligned} \tag{4.3}$$

then

$$W^{(\ell+1)}(z) =: \frac{\partial^\ell \phi^+(z)}{\partial z^\ell} \frac{\phi^-(z)}{z} : \tag{4.4}$$

and

$$\begin{aligned} W_k^{(\ell+1)} &= W_k(-\ell! \binom{D}{\ell}) \\ &= -t^k D(D-1)\cdots(D-\ell+1) + (-D)(-D-1)\cdots(-D-\ell+1)(-t)^k \\ &= -t^{k+\ell}(\frac{\partial}{\partial t})^\ell + (-)^{k+\ell} t (\frac{\partial}{\partial t})^\ell t^{k+\ell-1}. \end{aligned} \tag{4.5}$$

Using (4.2), we find that for $|w| < |y|$

$$W(y,w) = \frac{1}{2}\frac{y+w}{y-w}(X(y,w) - w^{-1}), \tag{4.6}$$

where $X(y,w)$ is the vertex operator defined in (1.3). Hence,

$$W^{(\ell)}(z) = \frac{w}{\ell}\frac{\partial^\ell X(y,z)}{\partial z^\ell}|_{y=z} + \frac{1}{2}\frac{\partial^{\ell-1} X(y,z) - z^{-1}}{\partial z^{\ell-1}}|_{y=z}. \tag{4.7}$$

Define

$$\alpha_j(z) = \begin{cases} \frac{1}{2}x & \text{if } j=-1, \\ -\frac{j}{2}t_{-j} & \text{if } j<2 \text{ odd}, \\ \frac{\partial}{\partial x} & \text{if } j=1, \\ \frac{\partial}{\partial t_j} & \text{if } j>2 \text{ odd}, \end{cases} \tag{4.8}$$

and their generating series by

$$\alpha(z) = \sum_{j\in\mathbf{Z}} \alpha_j z^{-j-1}, \tag{4.9}$$

then $[\alpha_j, \alpha_k] = \frac{j}{2}\delta_{j,-k}$. Since $X(z,z) = z^{-1}$, one finds the following expression for $W^{(\ell)}(z)$:

$$W^{(\ell)}(z) = \frac{2}{\ell} : (2\alpha(z) + \frac{\partial}{\partial z})^{\ell-1}\alpha(z) : + \frac{1}{z} : (2\alpha(z) + \frac{\partial}{\partial z})^{\ell-2}\alpha(z) : . \tag{4.10}$$

For $\ell = 1, 2, 3$ one finds respectively

$$\begin{aligned}
W^{(1)}(z) &= 2\alpha(z), \\
W^{(2)}(z) &= 2 : \alpha(z)^2 : + \frac{\partial\alpha(z)}{\partial z} + \frac{\alpha(z)}{z}, \\
W^{(3)}(z) &= \frac{8}{3} : \alpha(z)^3 : + \frac{8}{3} : \alpha(z)\frac{\partial\alpha(z)}{\partial z} : + \frac{2}{z} : \alpha(z)^2 : + \frac{\partial\alpha(z)}{\partial z} + \frac{2}{3}\frac{\partial^2\alpha(z)}{\partial z^2}.
\end{aligned}$$

5. The BKP hierarchy

5.1. The BKP hierarchy is the following equation for $\tau = \tau(t_1, t_3, \ldots)$ (see e.g [DJKM3], [K], [L2], [Y]):

$$\mathrm{Res}_{z=0}\frac{dz}{z}\phi^+(z)\tau \otimes \phi^-(z)\tau = \frac{1}{2}\theta\tau \otimes \theta\tau. \tag{5.1}$$

Here $\mathrm{Res}_{z=0}dz\sum_j f_j z^j = f_{-1}$. We assume that τ is any solution of (5.1), so we no longer assume that τ is a polynomial in $t_1, t_3, \ldots$.

We proceed now to rewrite (5.1) in terms of formal pseudo–differential operators. We start by multiplying (5.1) from the left with $\frac{\partial}{\partial\theta} \otimes \frac{\partial}{\partial\theta}$ and divide both the first and the last component of the tensor product by $\tau(t)$. Let $x = t_1$ and $\partial = \frac{\partial}{\partial x}$, then (5.1) is equivalent to the following bilinear identity:

$$\mathrm{Res}_{z=0}\frac{dz}{z}w(x, t, z)w(x', t', -z) = 1, \tag{5.2}$$

where

$$\begin{aligned}
w(x, t, \pm z) &= W(x, t, \pm z)e^{xz} = W(x, t, \partial)e^{xz} \qquad \text{with} \\
W(x, t, z) &= P(x, t, z)e^{\xi(t,z)}, \quad \xi(t, z) = \sum_{i\geq 3} t_i z^i \qquad \text{and}
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
P(x, t, z) &= \frac{e^{-\eta(x,t,z)}\tau(x, t)}{\tau(x, t)} \\
&= \frac{\tau(x_-\frac{2}{z}, t_3 - \frac{2}{3z^3}, t_5 - \frac{2}{5z^5}, \cdots)}{\tau(x, t)} =: \frac{\tilde{\tau}(x, t, z)}{\tau(x, t)},
\end{aligned} \tag{5.4}$$

where $\eta(x, t, z) = 2(\frac{\partial}{\partial x}z^{-1} + \sum_{j>2}\frac{\partial}{\partial t_j}\frac{z^{-j}}{j})$, for convenience we also define $\xi(x, t, z) = \xi(t, z)e^{xz}$.

5.2. As usual one denotes the differential part of a pseudo–differential operator $P = \sum_j P_j\partial^{-j}$ by $P_+ = \sum_{j\geq 0} P_j\partial^{-j}$ and writes $P_- = P - P_+$. The anti–involution $*$ is defined as follows $(\sum_j P_j\partial^{-j})^* = \sum_j(-\partial)^{-j}P_j$ One has the following fundamental lemma.

Lemma 5.1. *Let $P(x, t, \partial)$ and $Q(x, t, \partial)$ be two formal pseudo–differential operators, then*

$$(P(x, t, \partial)Q(x, t', \partial))_- = \pm\sum_{i>0} R_i(x, t, t')\partial^{-i}$$

if and only if

$$\mathrm{Res}_{z=0}dzP(x, t, \partial)e^{\pm xz}Q(x', t', \partial)e^{\mp x'z} = \sum_{i>0} R_i(x, t, t')\frac{(x - x)^{i-1}}{(i - 1)!}.$$

The proof of this lemma is analogous to the proof of Lemma 4.1 of [L1] (see also [KL]).

5.3. Now differentiate (5.2) to t_k, where we assume that $x = t_1$, then we obtain

$$\mathrm{Res}_{z=0}\frac{dz}{z}(\frac{\partial P(x, t, z)}{\partial t_k} + P(x, t, z)z^k)e^{\xi(x,t,z)}P(x't', -z)e^{-\xi(x',t',z)} = 0. \tag{5.5}$$

Now using lemma 5.1 we deduce that

$$((\frac{\partial P}{\partial t_k} + P\partial^k)\partial^{-1}P^*)_- = 0.$$

From the case $k = 1$ we then deduce that $P^* = \partial P^{-1}\partial^{-1}$, if $k \neq 1$, one thus obtains

$$\frac{\partial P}{\partial x_k} = -(P\partial^k P^{-1}\partial^{-1})_- \partial P. \tag{5.6}$$

Since k is odd, $\partial^{-}1(P\partial^k P^{-1})^*\partial = -P\partial^k P^{-1}$ and hence $(P\partial^k P^{-1}\partial^{-1})_-\partial = (P\partial^k P^{-1})_-$. So (5.6) turns into Sato's equation:

$$\frac{\partial P}{\partial t_k} = -(P\partial^k P^{-1})_- P. \tag{5.7}$$

5.4. Define the operators

$$\begin{aligned} L = W\partial W^{-1} = P\partial P^{-1}, &\quad \Gamma = x + \sum_{j>2} j t_j \partial^{j-1}, \\ M = &W x W^{-1} = P\Gamma P^{-1} \quad \text{and } N = ML. \end{aligned} \tag{5.8}$$

Then $[L, M] = 1$ and $[L, N] = L$. Let $B_k = (L^k)_+$, using (5.7) one deduces the following Lax equations:

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial M}{\partial t_k} = [B_k, M] \quad \text{and } \frac{\partial N}{\partial t_k} = [B_k, N]. \tag{5.9}$$

The first equation of (5.9) is equivalent to the following Zakharov Shabat equation:

$$\frac{\partial B_j}{\partial t_k} - \frac{\partial B_k}{\partial t_j} = [B_k, B_j], \tag{5.10}$$

which are the compatibility conditions of the following linear problem for $w = w(x, t, z)$:

$$Lw = zw, \quad Mw = \frac{\partial w}{\partial z} \quad \text{and} \quad \frac{\partial w}{\partial t_k} = B_k w. \tag{5.11}$$

5.5. The formal adjoint of the wave function w is (see [DJKM]):

$$w^* = w^*(x, z) = P^{*-1}e^{-\xi(x,t,z)} = \partial P \partial^{-1} e^{-\xi(x,t,z)}. \tag{5.12}$$

Now $L^* = -\partial L \partial^{-1} = -\partial P \partial P^{-1}\partial^{-1}$ and $M^* = \partial P \partial^{-1}\Gamma \partial P^{-1}\partial^{-1}$, so $[L^*, M^*] = -1$ and

$$L^* w^* = z w^*, \quad M^* w^* = -\frac{\partial w^*}{\partial z} \quad \text{and} \quad \frac{\partial w^*}{\partial x_k} = -(L^{*k})_+ w^* = -B_k^* w^*. \tag{5.13}$$

Finally, notice that by differentiating the bilinear identity (5.2) to x'_1 we obtain

$$\mathrm{Res}_{z=0} dz w(x, t, z) w^*(x', t', z) = 0. \tag{5.14}$$

6. Proof of Theorem 1

6.1. In this section we prove Theorem 1.1. We start from the bilinear identity (5.14) and multiply it by $\tau(x,t)$, which gives

$$\mathrm{Res}_{z=0} dz e^{-\eta(x,t,z)}\tau(x,t)e^{\xi(x,t,z)}\frac{\partial}{\partial x'}\left(\frac{e^{\eta(x',t',z)}\tau(x',t')}{\tau(x',t')}e^{-\xi(x',t',z)}\right)=0. \tag{6.1}$$

Now let $(1-w/y)^{-1}(1+w/y)X(y,w)$ act on this identity, then one obtains

$$\begin{aligned}&\mathrm{Res}_{z=0}\frac{dz}{wz}\frac{1+w/y}{1-w/y}\frac{1-z/y}{1+z/y}\frac{1+z/w}{1-z/w}e^{-\eta(x,t,z)-\eta(x,t,y)+\eta(x,t,w)}\tau(x,t)e^{\xi(x,t,z)+\xi(x,t,y)-\xi(x,t,w)}\times\\ &\quad\frac{\partial}{\partial x'}\left(\frac{e^{\eta(x',t',z)}\tau(x',t')}{\tau(x',t')}e^{-\xi(x',t',z)}\right)=0.\end{aligned} \tag{6.2}$$

Next use the fact that $(1-u)^{-1}(1+u)=2\delta(u,1)-(1-u^{-1})^{-1}(1+u^{-1})$, where $\delta(u,v)=\sum_{j\in\mathbf{Z}}u^{-j}v^{j-1}$, then (6.2) is equivalent to

$$\begin{aligned}&-\mathrm{Res}_{z=0}\frac{dz}{wz}\frac{1+w/y}{1-w/y}\frac{1-y/z}{1+y/z}\frac{1+w/z}{1-w/z}e^{-\eta(x,t,z)-\eta(x,t,y)+\eta(x,t,w)}\tau(x,t)e^{\xi(x,t,z)+\xi(x,t,y)-\xi(x,t,w)}\times\\ &\quad\frac{\partial}{\partial x'}\left(\frac{e^{\eta(x',t',z)}\tau(x',t')}{\tau(x',t')}e^{-\xi(x',t',z)}\right)=\frac{2}{w}(e^{-\eta(x,t,y)}\tau(x,t)e^{\xi(x,t,y)}\frac{\partial}{\partial x'}\left(\frac{e^{\eta(x',t',w)}\tau(x',t')e^{-\xi(x',t',w)}}{\tau(x',t')}\right)\\ &\quad-e^{\eta(x,t,w)}\tau(x,t)e^{-\xi(x,t,w)}\frac{\partial}{\partial x'}\left(\frac{e^{-\eta(x',t',y)}\tau(x',t')}{\tau(x',t')}e^{\xi(x',t',y)}\right)).\end{aligned} \tag{6.3}$$

Divide this formula by $\tau(x,t)$, then it turns into

$$\begin{aligned}&-\mathrm{Res}_{z=0}\frac{dz}{z}e^{-\eta(x,t,z)}\left(\frac{1+w/y}{1-w/y}\frac{X(y,w)\tau(x,t)}{\tau(x,t)}\right)\frac{e^{-\eta(x,t,z)}\tau(x,t)}{\tau(x,t)}e^{\xi(x,t,z)}\frac{\partial}{\partial x'}\left(\frac{e^{\eta(x',t',z)}\tau(x',t')e^{-\xi(x',t',z)}}{\tau(x',t')}\right)\\ &=2\mathrm{Res}_{z=0}\frac{dz}{z}\delta(w,z)(\frac{e^{-\eta(x,t,y)}\tau(x,t)}{\tau(x,t)}e^{\xi(x,t,y)}\frac{\partial}{\partial x'}\left(\frac{e^{\eta(x',t',z)}\tau(x',t')}{\tau(x',t')}e^{-\xi(x',t',z)}\right)\\ &\quad-\frac{e^{\eta(x,t,z)}\tau(x,t)}{\tau(x,t)}e^{-\xi(x,t,z)}\frac{\partial}{\partial x'}\left(\frac{e^{-\eta(x',t',y)}\tau(x',t')}{\tau(x',t')}e^{\xi(x',t',y)}\right))\\ &=2\mathrm{Res}_{z=0}\frac{dz}{w^2}\sum_{\ell=0}^{\infty}\frac{(y-w)^\ell}{\ell!}\sum_{k\in\mathbf{Z}}((\frac{z}{w})^{k+\ell-1}(\frac{\partial}{\partial z})^\ell(W(x,t,z)e^{xz})\frac{\partial}{\partial x'}(W(x',t',-z)e^{-x'z})\\ &\quad-W(x,t,-z)e^{-xz}(\frac{z}{w})^{k+\ell-1}(\frac{\partial}{\partial z})^\ell\frac{\partial}{\partial x'}(W(x',t',z)e^{x'z}))\\ &=2\mathrm{Res}_{z=0}dz\sum_{\ell=0}^{\infty}\frac{(y-w)^\ell}{\ell!}\sum_{k\in\mathbf{Z}}w^{-k-\ell-1}(W(x,t,z)x^\ell\partial^{k+\ell-1}e^{xz}\frac{\partial}{\partial x'}(W(x',t',-z)e^{-x'z})\\ &\quad-W(x,t,-z)e^{-xz}\frac{\partial}{\partial x'}(W(x',t',z)x^\ell\partial^{k+\ell-1}e^{x'z})).\end{aligned} \tag{6.4}$$

Now define

$$\sum_{j=0}^{\infty}c_j(x,t,y,w)z^{-j}=e^{-\eta(x,t,z)}\left(\frac{1+w/y}{1-w/y}\frac{X(y,w)\tau(x,t)}{\tau(x,t)}\right), \tag{6.5}$$

then the first line of (6.4) is equal to

$$-\mathrm{Res}_{z=0}dz\sum_{j=0}^{\infty}c_j(x,t,y,w)L^{-j-1}(W(x,t,z)e^{xz})\frac{\partial}{\partial x'}(W(x',t',-z)e^{-x'z}).$$

Now using Lemma 5.1 with $t=t'$, one deduces that

$$\frac{1}{2}\sum_{j=1}^{\infty}c_j(x,t,y,w)L^{-j}=$$
$$-\sum_{\ell=0}^{\infty}\frac{(y-w)^\ell}{\ell!}\sum_{k\in\mathbf{Z}}w^{-k-\ell-1}(W(x,t,z)x^\ell\partial^{k+\ell}W(x,t,z)^{-1}-(-)^{k+\ell}W(x,t,z)\partial^{k+\ell-1}x^\ell\partial W(x,t,z)^{-1})_-. \tag{6.6}$$

So finally one has

$$\frac{1}{2}(e^{-\eta(x,t,z)}-1)\left(\frac{1+w/y}{1-w/y}\frac{X(y,w)\tau(x,t)}{\tau(x,t)}\right)w(x,t,z)=$$
$$-\sum_{\ell=0}^{\infty}\frac{(y-w)^\ell}{\ell!}\sum_{k\in\mathbf{Z}}w^{-k-\ell-1}(M^\ell L^{k+\ell}-(-)^{k+\ell}L^{k+\ell-1}M^\ell L)_-w(x,t,z),$$

wich is equal to the Adler–Shiota–van Moerbeke formula (1.2) for the BKP case:

$$(w+y)(e^{-\eta(x,t,z)}-1)\left(\frac{X(y,w)\tau(x,t)}{\tau(x,t)}\right)w(x,t,z)=2(w-y)Y(y,w)_-w(x,t,z).$$

6.2. Since the left–hand–side of (1.2) is also equal to

$$(w+y)(e^{-\eta(x,t,z)}-1)\left(\frac{(X(y,w)-1)\tau(x,t)}{\tau(x,t)}\right)w(x,t,z),$$

we have the following corollary of Theorem 1.1:

Corollary 6.1. *For $k\in\mathbf{Z}$ and f some polynomial one has*

$$\frac{\sigma\cdot\hat{\pi}\cdot\rho(W_k(f))\tau(x,t)}{\tau(x,t)}=\frac{(f(N)L^k-(-L)^kf(-N))_-w(x,t,z)}{w(x,t,z)}. \tag{6.7}$$

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